A Multiple Input Image Restoration Approach

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In this paper image restoration applications, where multiple distorted versions of the same original image are available, are considered. A general adaptive restoration algorithm is derived on the basis of a set theoretic regularization technique. The adaptivity of the algorithm is introduced in two ways: (a) by a constraint operator which incorporates properties of the response of the human visual system into the restoration process and (b) by a weight matrix which assigns greater importance for the deconvolution process to areas of high spatial activity than to areas of low spatial activity. Different degrees of trust are assigned to the various distorted images depending on the amounts of noise. The proposed algorithm is general and can be used for any type of linear distortion and constraint operators. It can also be used to restore signals other than images. Experimental results obtained by an iterative implementation of the proposed algorithms are presented. © 1990 Academic Press, Inc.

1. INTRODUCTION

The recovery or restoration of an image that has been distorted is one of the most important problems in image processing [2]. It is probably safe to say that there is no field where images are acquired that does not have active or potential work in image restoration. In this paper we consider image restoration applications where multiple distorted versions of the same original image are available. Such applications include, for example, the simultaneous exposure of a moving object by a number of independent cameras (Sandia National Laboratories [6]) and the acquisition of differently defocused images of the same specimen in electron microscopy [22]. As another application we mention the restoration of sequences of images, such as video images. In this case, although the frames in the sequence differ due to moving objects, the frames become identical if we assume that perfect compensation for the motion can be achieved.

More specifically, it is assumed that \( m \) distorted versions \( y_i, i = 1, \ldots, m \), of the original image \( x \) are available, where each of the \( y_i \)'s and \( x \) are represented by \( N \times 1 \) vectors. That is, for lexicographically ordered images it holds that

\[
y_i = D_i x + n_i, \quad i = 1, \ldots, m,
\]

where the \( N \times N \) matrices \( D_i \) represent the known deterministic (space varying in general) distortion and \( n_i \) represents the additive noise. The image restoration problem then is stated as follows: given the \( y_i \)'s, \( i = 1, \ldots, m \), find an image as close as possible to the original image, subject to a suitable optimality criterion. We have presented an iterative approach for solving Eq. (1) when the noise is ignored [11] and later an adaptive iterative approach when the noise is taken into account [14]. Ghiglia [6] has also presented a nonadaptive constrained least-squares algorithm formulated in the discrete frequency domain. One of the disadvantages in Ghiglia's approach is that the different observations \( y_i \) are equally weighted independently of the amount of noise corresponding to each observation.

Clearly, as with the restoration of single images, different restoration algorithms can be obtained depending on the model used for the original image \( x \). In this paper we obtain solutions to Eq. (1) by treating \( x \) as a deterministic signal. In these solutions a different weight or different "degree of trust" is assigned to the various observations depending on their amount of noise. We also address the question of distributed or parallel restoration. We show for one formulation of the problem that the solution obtained by processing all the available images simultaneously is also obtained if each image \( y_i \) is restored independently to produce \( x_i \), and then these solutions are appropriately combined to produce the final answer.

The paper is organized as follows. In Section 2 we obtain solutions to Eq. (1) on the basis of a set theoretic regularization approach. The method of projecting onto convex sets (POCS) and a method that considers an ellipsoid which bounds the set of feasible solutions are employed for determining the restored image. In Section 3 ways to choose the weight matrices which introduce the spatial adaptivity are proposed. In Section 4 the distrib-
uted or parallel restoration of the available distorted images is considered, when no prior knowledge about the original image is available. The iterative implementation of the proposed algorithms is described in Section 5. Finally experimental results are shown in Section 6 and conclusions are presented in Section 7.

2. SET THEORETIC RESTORATION

Let us initially consider the independent restoration of each frame, or equivalently, let us set $m = 1$ in Eq. (1). If the image formation process is modeled in a continuous infinite dimensional space each $D_i$ becomes an integral operator and each of the equations in (1) becomes a Fredholm integral equation of the first kind. Then the solution of (1) is almost always an ill-posed problem [17, 19, 21]. This means that the unique least-squares solution of minimal norm of (1) does not depend continuously on the data, that a bounded perturbation (noise) in the data results in an unbounded perturbation in the solution, or that the generalized inverse of each $D_i$ is unbounded [19]. Each integral operator $D_i$ has a countably infinite number of singular values that can be ordered with their limit approaching zero [19]. Since the finite dimensional discrete problem of image restoration results from the discretization of an ill-posed continuous problem, each $D_i$ has (in addition to possibly a number of zero singular values) a cluster of very small singular values. Clearly, the finer the discretization (the larger the sizes of $D_i$), the closer the limit of the singular values is approximated. Therefore, although the finite dimensional inverse problem is well-posed in the least-squares sense [19], the ill-posedness of the continuous problem translates into ill-conditioned matrices $D_i$. The ill-conditionedness of each of the $D_i$’s will most probably result in an ill-conditioned composite matrix, depending on $m$ and the spectral properties of the $D_i$’s, as explained in Section 2.3.

A regularization method replaces an ill-posed problem by a well-posed problem, whose solution is an acceptable approximation to the solution of the given ill-posed problem [26]. It defines a solution by trying to achieve smoothness and yet remain “faithful” to the data. In other words, a regularized solution is a solution between the “ultrarough” least-squares solution and an “ultrasmooth” solution based on a priori knowledge [4, 27]. The incorporation of a priori knowledge into the restoration is the basic underlying idea in most regularization approaches. A set theoretic regularization approach [24] is followed in this work, as presented next.

2.1. Formulation

According to a set theoretic approach a priori knowledge is available about $x$, which restricts it to lie in a set, that is,

$$x \in Q_x.$$  (2)

where $Q_x$ is a set in an $N$-dimensional space. Similarly, assume that the noise $n_i$ belongs to a set $Q_n$. Since $n_i$ must lie in a set, it follows that a given observation $y_i$ combines with the set $Q_n$ to define a new set which must contain $x$. Thus the observation $y_i$ specifies a set $Q_{xy}$, which must contain $x$; i.e.,

$$x \in Q_{xy} = \{ x : (y_i - D_i x) \in Q_n \}, \quad i = 1, \ldots, m.$$  (3)

Consider now the sets $Q_x$, $Q_{xy}$, . . . , $Q_{xy}$. Each set contains $x$, and therefore, $x$ must lie in their intersection, denoted by $Q_x$. Then

$$Q_0 = Q_x \cap Q_{xy} \cap \cdots \cap Q_{xy},$$  (4)

where $\cap$ denotes set intersection. According to Eq. (4) the restored image is defined as a set. This set is the smallest set which must contain $x$ and which can be calculated from the available information. We note here that the formulation of this set theoretic approach to restoration is quite straightforward and holds true for any kind of sets. However, the difficulty of the approach arises when the intersection of these sets, as defined by Eq. (4), is to be calculated. In making the problem more tractable, ellipsoids are used for the sets $Q_x$ and $Q_n$.

The equation of an ellipsoid is given by

$$Q_x = \{ x : (x - c_x)^T \Gamma^{-1} (x - c_x) \leq 1 \},$$  (5)

where $c_x$ is the center of the ellipsoid and $\Gamma$ is a positive definite matrix, whose eigenvectors and eigenvalues determine respectively the orientation and the lengths of the axis of $Q_x$. For the image restoration problem, the usual form of $Q_x$ is given by [27].

$$\|Cx\|^2 \leq E^2,$$  (6)

where $\| \cdot \|_F$ represents a weighted norm. The form of the matrices $F$ and $C$ and the value of $E$ are discussed later. Clearly, Eq. (6) represents an ellipsoid with $c_x = 0$ and $\Gamma^{-1} = C^T F^T F C / E^2$. Similarly, a common way for enforcing $Q_n$ is by

$$Q_n = \{ n_i : \| n_i \|^2 \leq \sigma_i^2 \}, \quad i = 1, \ldots, m,$$  (7)

where $\sigma_i^2$ represents the noise variance and the form of the matrix $W$ is discussed later. Given (7), it is shown in Appendix 1 that each of the $Q_{xy}$’s is equal to

$$Q_{xy} = \{ x : (x - x_i^*)^T S^{-1} (x - x_i^*) \leq 1 \}, \quad i = 1, \ldots, m,$$  (8)
where \( S^{-1} = (1/e^2) D_i^T W_i^T W D_i \), \( x_i^* = D_i^T y_i \), and \( D_i^T \) denotes the generalized inverse of \( D_i \). Figure 1 shows the case of three two-dimensional intersecting ellipsoids, namely \( Q_0 \), \( Q_{\delta_1} \), and \( Q_{\delta_2} \). Their intersection is denoted by \( Q_0 \). Next, two general approaches for evaluating an image which belongs in the intersection \( Q_0 \) are investigated.

2.2. Projection onto Convex Sets

The method of POCS provides an iterative approach for determining a signal in the intersection of convex sets, provided that this intersection is not empty [29]. More specifically, let \( Q_1, Q_2, \ldots, Q_m \) be closed convex sets in a finite dimensional vector space, with \( P_1, P_2, \ldots, P_m \) their respective projectors. Then the iterative procedure

\[
x_{k+1} = P_1 P_2 \cdots P_m x_k
\]

converges to a vector that belongs in the intersection of the closed convex sets, for any starting vector \( x_0 \). It is noted that the resulting set intersection is also a closed convex set.

It is straightforward to show that the projection \( P_x x \) onto the set \( Q_x \) in Eq. (6) is given by [15]

\[
P_x x = [I - \lambda (I + \lambda C^T C)^{-1} C^T C] x,
\]

where \( \lambda \) needs to be chosen so that condition (6) is satisfied. Similarly, the projections \( P_i x \) onto the sets \( Q_{\delta_i} \) in Eq. (8) are given by [15, 28]

\[
P_i x = x + \lambda (I + \lambda D_i^T D_i)^{-1} D_i^T (y_i - D_i x), \quad i = 1, \ldots, m,
\]

where the \( \lambda_i \)'s need to be chosen so that condition (8) is satisfied. Any vector belonging on the boundary of \( Q_0 \) can be obtained as a fixed point of the POCS restoration process that vided that \( x_0 \not\in Q_0 \), where \( x_0 \) is the initial estimate of \( x \). If \( x_0 \in Q_0 \), then \( x_0 \) will be the solution. Clearly additional closed convex sets, which express properties of the original image, can be used in the formulation of the problem. An example of such a set is the set of all images with positive pixel values. If, on the other hand, no information is available about the original image \( x \), set \( Q_0 \) should be removed. Then the projections of Eq. (11) should be alternated in obtaining a solution in the intersection of the \( Q_{\delta_i} \)'s. Such an intersection is represented by \( Q_0 \cup \hat{Q}_0 \) in Fig. 1, where \( \cup \) denotes set union.

2.3. Use of Bounding Ellipsoids

Unfortunately, the intersection of ellipsoids is not necessarily an ellipsoid, as can also be seen in Fig. 1. Therefore, one approach to geometrically describing the intersection is to consider an ellipsoid which bounds it [24]. Then the center of this bounding ellipsoid is chosen to represent the restored image. The equations defining such an ellipsoid are shown in Appendix II. Given the ellipsoids of Eqs. (6) and (7), the center of a bounding ellipsoid is described by

\[
\left[ \sum_{i=1}^{m} \frac{\rho_i}{e_i^2} D_i^T W_i^T W D_i + \frac{\rho_{m+1}}{E^2} C^T F^T F C \right] x
\]

\[= \sum_{i=1}^{m} \frac{\rho_i}{e_i^2} D_i^T W_i^T W y_i, \quad (12)
\]

where \( \sum_{i=1}^{m+1} \rho_i = 1 \) and \( \rho_i > 0 \). For each choice of the values for the parameters \( \rho_i \), a different bounding ellipsoid whose center corresponds to a different estimate of the original image is defined. The center of one of the bounding ellipsoids, denoted by \( x_c \), is shown in Fig. 1. If all the \( \rho_i \)'s are chosen to be equal then Eq. (12) takes the form

\[
\left[ \sum_{i=1}^{m} e_i^{-2} D_i^T W_i^T W D_i + E^{-2} C^T F^T F C \right] x = \sum_{i=1}^{m} e_i^{-2} D_i^T W_i^T W y_i,
\]

Equation (13) confirms our intuition, according to which the contribution of each observation to the solution is inversely proportional to the amount of noise present. In other words, the more noisy observations are trusted less.

Equation (13) can also be derived by considering an extension of Miller's regularization approach [12, 15, 18]. According to it, a vector that satisfies

\[
\| C x \|_F^2 \leq E^2
\]

(14)
\[ \| y_i - D_i x \|_W \leq e_i^2, \quad i = 1, \ldots, m \]  

is sought.

One approach to finding an \( x \) that satisfies the above constraints is by combining these constraints quadratically. That is, a vector \( x \) which minimizes

\[ M(x) = E^{-2} \| Cx \|_F^2 + \sum_{i=1}^{m} e_i^{-2} \| y_i - D_i x \|_W^2 \]  

is sought.

The necessary condition for \( M(x) \) to have a minimum is that its gradient with respect to \( x \) be equal to zero. This condition leads to Eq. (13), as can be easily shown. It is mentioned here that the relation \( M(x) \leq (m + 1) \) defines an ellipsoid which bounds the intersection \( Q_\theta \) in Eq. (4), when \( \rho_i = 1/(m + 1), \ i = 1, \ldots, (m + 1) \). Therefore, Miller’s solution represents the center of one of the bounding ellipsoids.

Two special cases of the result presented above occur when no estimate for one of the bounds \( e_i \) or \( E \) is available and when \( E \) becomes infinite. In the first case, the prior knowledge defines all but one of the ellipsoids \( Q_i \) or \( Q_i^0 \). Then the constrained least-squares (CLS) approach [3, 8, 14] results in a restored image which is on the surface of the ellipsoids which are defined at the point of minimum distance from the center of the ellipsoid which is not defined. Two of these solutions, denoted by \( x_k \), when \( E \) is unknown, and \( x_k \), when \( e_k \) is known, are shown in Fig. 1. As can be seen in Fig. 1, \( x_k \) is inside both ellipsoids \( Q_i^0 \), and \( Q_i^0 \). Therefore, \( x_k \) is closer to both centers of the two ellipsoids, which represent ultrarough solutions. By examining Fig. 1 it is clear that, by reducing the value of \( E \), the size of the ellipsoid \( Q_i \) reduces, and the solution \( x_k \) moves closer to \( x_k \). These properties of the solutions have been verified experimentally.

In the second case, the set \( Q_i \) does not enter the solution, since any vector will belong to it. Then Eq. (13) reduces to

\[ \left( \sum_{i=1}^{m} e_i^{-2} D_i^T W_i D_i \right) x = \sum_{i=1}^{m} e_i^{-2} D_i^T W_i y_i. \]  

Equation (17) results also from a regression approach to the restoration problem at hand. Such an approach is described in detail in Section 4, since it also provides a solution to an independent or distributed restoration scenario.

2.4. Geometric Characterization of the Intersection

According to Eq. (12) only the ratio of the parameters \( e_i \) and \( E \) is required in obtaining a solution. However, the individual values of \( e_i \) and \( E \) are needed in verifying the existence of a feasible set of solutions or the nonemptiness of \( Q_\theta \). Clearly, the same ratio \( (e_i/E) \) results by multiplying both parameters by the same factor \( d \). For certain values of \( d \) the sets \( Q_i \) and \( Q_i^0 \) will not intersect. Then \( Q_i \) is empty and the solution \( x_k \) obtained from (12) does not exhibit the a priori known properties of the original image. The nonemptiness of \( Q_i \) is checked by considering the matrix \( \Gamma \) of the bounding ellipsoid (Appendix II). It is equal to

\[ \Gamma_i^{-1} = \left[ \begin{array}{c} e_i^2 \\ \rho_i \end{array} \right] \left( \begin{array}{c} \sum_{i=1}^{m} \alpha_i (\| y_i \|_W^2 - (y_i, D_i x_k)_W) \\ \sum_{i=1}^{m} \alpha_i D_i^T W_i D_i + \alpha_{m+1} C_i F_i T F_i C_i \end{array} \right) \]

where \( \alpha_i = (\rho_i e_i^2) \cdot (e_i^2/\rho_i), \ i = 1, \ldots, m; \ \alpha_{m+1} = (\rho_{m+1} e_{m+1}^2)/E^2 \cdot (e_{m+1}^2/\rho_{m+1}); \ \alpha_i (a, b)_W = a^T W_i b; \) and \( x_k \) satisfies Eq. (12). If \( \delta_i(e_i, E, \rho_i, y_i, x_k) \geq 0 \), then \( Q_i \) and the \( Q_i^0 \) intersect, which means that the prior knowledge is consistent. Since \( \delta_i(e_i, E, \rho_i, y_i, x_k) \) is a function of \( x_k \), we have only an an a posteriori test of the consistency of the prior knowledge. It is straightforward to show that for \( \rho_i = 1/(m + 1) \) the condition \( \delta_i(e_i, E, \rho_i, y_i, x_k) \geq 0 \) is equivalent to the condition \( M(x) \leq (m + 1) \), where \( M(x) \) is shown in Eq. (16).

The size of \( \Gamma_i^{-1} \) can be used as an estimate of the difference among feasible solutions. A measure of the size of \( \Gamma_i^{-1} \) can be obtained by considering the largest distance between two vectors in \( Q_i \). It is equal to the maximum length of the major axes denoted by \( \delta(e_i, E, \rho_i, y_i, x_k) \). According to Eq. (18) it is equal to

\[ \delta(e_i, E, \rho_i, y_i, x_k) = \frac{1}{\lambda(e_i, E, \rho_i)} \delta_i(e_i, E, \rho_i, y_i, x_k)^{-1/2}, \]  

where \( \lambda^2(e_i, E, \rho_i) \) is the smallest eigenvalue of \( \{ \sum_{i=1}^{m} \alpha_i D_i^T W_i D_i + \alpha_{m+1} C_i F_i T F_i C_i \} \), and \( \rho_i e_i^2 \) is the largest of all \( \rho_i e_i^2, \ i = 1, \ldots, m \), factors. The measure of Eq. (19) depends on the data, and therefore it cannot be precomputed. A looser measure of the size of \( \Gamma_i^{-1} \), independent of the data, is given by

\[ \delta(e_i, E, \rho_i) = \frac{e_i \sqrt{\rho_i}}{\lambda(e_i, E, \rho_i)} \]  

since \( \delta(e_i, E, \rho_i, y_i, x_k) \leq \delta(e_i, E, \rho_i) \), as shown in Appendix III. Clearly the size of \( \Gamma_i^{-1} \) should be minimized.
However, $\Gamma_0^{-1}$ is not only a function of $\rho_0$, but also a function of the data. Such a minimization even with respect to one distorted image is complicated, since $x_0$ also depends on $\rho_0$. Therefore, in our experiments the values $\rho_i = \frac{1}{(m + 1)}$ were used.

3. CHOICE OF THE CONSTRAINT OPERATOR AND THE WEIGHT MATRICES

3.1. Properties and Choice of $C$

The matrix $C$ in (6) should be chosen to describe some known properties of the original signal while rendering the resulting system of linear equations (13) better conditioned than the system of equations in Eq. (17). That is, $C$ should be chosen in such a way that the $\mathcal{P}$-condition numbers satisfy

$$
\mathcal{P}\left[ E^{-2}C^T F^T F C + \sum_{i=1}^{m} \varepsilon_i^{-2} D_i^T W^T W D_i \right] < \mathcal{P}\left[ \sum_{i=1}^{m} \varepsilon_i^{-2} D_i^T W^T W D_i \right], \quad (21)
$$

where $\mathcal{P}(A) = \|A\|_1 \cdot \|A^+\|_2$ and $A^+$ denotes the generalized inverse of $A$. Conceptually, what inequality (21) tells us is that it would be desirable that the constraint matrix $C^T C$ leave the large eigenvalues of $K^T K = [\sum_{i=1}^{m} \varepsilon_i^{-2} D_i^T W^T W D_i]$ unchanged, while it moves the small eigenvalues of $K^T K$ away from zero, without introducing new small eigenvalues. According to this, $C^T C$ should be a singular matrix so that the large eigenvalues of $K^T K$ are not altered. Therefore, if $C^T C$ is chosen this way, $r'$, the dimensionality of the range of $(E^{-2}C^T F^T F C + K^T K)$, will be at least as large as $r$, the dimensionality of the range of $K^T K$.

Condition (21) is easily verified if $F = W = I$ and if we assume that the matrices $K^T K$ and $C^T C$ commute, which means that the two matrices have the same complete set of eigenvalues [25]. Then

$$
\mathcal{P}(E^{-2}C^T C + K^T K) = \frac{\max_j (E^{-2}\sigma_j^2) + \sum_{i=1}^{m} \varepsilon_i^{-2} \mu_{i,j}^2}{\min_j (E^{-2}\sigma_j^2) + \sum_{i=1}^{m} \varepsilon_i^{-2} \mu_{i,j}^2}, \quad (22)
$$

where $\mu_{i,j}$ and $\sigma_j$ are respectively the singular values of $D_i$, $i = 1, \ldots, m$, and $C$, and $\mu_{i,1} \geq \mu_{i,2} \geq \cdots \geq \mu_{i,m} \geq \mu_{i,m+1} = \cdots = 0$. The assumption that the matrices $K^T K$ and $C^T C$ commute may be somewhat restrictive, although it holds for a large class of problems of interest, namely when each of the $D_i$'s and $C$ represent linear space-invariant systems. In this case, the matrices $K^T K$ and $C^T C$ can be represented by block circulant matrices and Eq. (13) takes the form

$$
(E^{-2}|C(v, v)|^2 + \sum_{i=1}^{m} \varepsilon_i^{-2}|D_i(v, v)|^2) X(v, v) = \sum_{i=1}^{m} \varepsilon_i^{-2} D_i^T(v, v) Y_i(v, v), \quad (23)
$$

for $\nu = 0, \ldots, N - 1$ and $v = 0, \ldots, N - 1$. In Eq. (23) $D_i(v, v)$ and $C(v, v)$ represent the frequency responses of the systems that introduce the distortion and the constraint; $X(v, v)$ and $Y_i(v, v)$ are the 2D DFTs of the original and distorted images, respectively; and * denotes complex conjugate. From Eq. (23) the desirable properties of the constraint become clear. That is, since each $\mu_{i,j}$ is a decreasing sequence with $j$, $\sigma_j$ should be an increasing sequence with $j$. In other words, if each of the $D_i(v, v)$'s is the frequency response of a low-pass filter (a very common type of distortion), then $C(v, v)$ must be the frequency response of a high-pass filter. Conceptually, the constraint $C$ is chosen so that the energy of the restored signal at high frequencies, mainly due to the amplification of the broadband noise, is suppressed. In general, if $\sum_{i=1}^{m} |D_i(v, v)|^2$ is a bandpass filter, $|C(v, v)|^2$ must be a bandstop filter, with their stop- and passbands interchanged.

In general, since $C^T C$ can be chosen by the designer, he/she can require that it have the same eigenvectors with each of the $D_i^T D_i$'s. However, if the $D_i^T D_i$'s do not have a special structure, such as the block Toeplitz structure, the determination of the eigenvalues and eigenvectors of a large matrix may be a computationally difficult problem. Therefore, if the above analysis is difficult or if it is chosen to be avoided, a smoothness requirement on the solution can be imposed by requiring that $C$ be a high-pass filter. In agreement with this, $C$ was chosen to be a $p$th-order differential operator (high-pass filter) when the CLS method was implemented [8, 21, 26].

It is important to note at this point that although the incorporation of $C$ is necessary for the restoration of noisy–blurred single frames with low SNRs, $C$ may not be necessary for the multiple input restoration problem. This is due to the fact that the condition of the composite matrix $K^T K$ improves in most cases of interest, when compared to the condition of each of the $D_i^T D_i$'s. That is, it holds that $\mathcal{P}(K^T K) < \mathcal{P}(D_i^T D_i)$, for all $i$. This condition is easily verified for the case of space-invariant distortions. Then zeros in the frequency response of one $D_i$ are removed by values of the frequency response of another $D_i$.

3.2. Choice of $F$ and $W$

With the matrices $F$ and $W$ the spatial adaptivity of the restoration filter is introduced, on the basis of properties of the human visual system. The adaptivity of the restoration filter is necessary since the mean-squared-error-
based restoration filter is low-pass and gives rise to unacceptable blurring of lines and edges in the image. Psychophysical experiments confirm that the human visual system is characterized by a noise masking effect, which results in lower noise visibility in the vicinity of edges. On the basis of this information, Anderson and Netravali [1] first defined the noise masking function at each pixel location as a measure of spatial detail. Then, they performed subjective tests and obtained the visibility function at each pixel location, which expresses the relationship between the visibility of noise and the masking function.

We have proposed and used the local variance as a measure of the spatial detail [13–15, 17]. Following [1] we also defined the visibility function \( f(i,j) \) as

\[
f(i,j) = \frac{1}{\theta \cdot \sigma^2(i,j) + 1},
\]

where \( \sigma^2(i,j) \) is the local variance of \( x \) and \( \theta \) is a tuning parameter that must be adjusted experimentally for each class of images. The visibility function is normalized and takes values between zero and one. It is clear from Eq. (24) that for the areas with high spatial activity (large value of \( \sigma^2(i,j) \)) the visibility function goes to zero (noise is not visible), while for flat areas (small \( \sigma^2(i,j) \)) the visibility function goes to one (noise is visible).

The diagonal matrix \( F \) has the entries stacked along the diagonal values of \( f(i,j) \) in Eq. (24). Then \( W \) is set to be equal to \( I - F \). This way at the areas of high spatial activity the noise is allowed to go through the restoration filter, due to the action of \( F \), while these areas are more deblurred than the areas of low spatial activity, due to the action of \( W \). Ichioka and Nakajima [10] have also used a \( W \) in their iterative single image frame restoration algorithm, which they called the nonlinear constraint matrix.

In applying the adaptive algorithm a good estimate of the spatial activity of the image, as expressed by the visibility function, is required. However, since only the noisy–blurred image is available, this estimate of the visibility function will be in error, and therefore errors will be introduced in the restored image. Two ways to solve this problem are: (a) Obtain a restored image first with the use of any restoration algorithm, compute \( F \), and then run the adaptive algorithm. (b) Since the iterative algorithm presented in Section 5 is used in our work, \( F \) is estimated at each iteration on the basis of the available form of the restored image. Although the convergence analysis of the algorithm has not been established in this case, the algorithm has converged in all the experiments.

4. INDEPENDENT RESTORATION

The case when the bound \( E \) in the formulation of the previous section becomes infinite is considered in this section. Such a case is of interest because the constraint \( C \) may not be required, depending on the value of \( m \) and the condition of each of the \( D_i \)'s as was mentioned earlier. Equation (1) is rewritten as

\[
y = Dx + n, \tag{25}
\]

where \( y \) and \( n \) are formed by stacking the vectors \( y_i \) and \( n_i \), respectively, and \( D = (D_1 \cdots D_m)'. \) Since in Eq. (1) each of the vectors is of size \( N \times 1 \), the vectors \( y \) and \( n \) in Eq. (25) are of size \( mN \times 1 \) and the matrix \( D \) is of size \( mN \times N \). It is assumed that

\[
E[n_i] = 0 \quad \text{and} \quad E[n_i n_i'] = R_i, \quad i = 1, \ldots, m. \tag{26}
\]

Then, assuming that the noise processes are uncorrelated, it holds that \( E[n_i n_i'] = R_i \), where \( R \) is a block diagonal matrix with each block equal to \( R_i \). Given Eq. (25) a regression approach provides a linear unbiased estimator which yields the minimum error covariance matrix, and the resulting restored image is equal to [7]

\[
x = [D_i^T R_i^{-1}D_i]^{-1}D_i^T R_i^{-1}y, \tag{27}
\]

or

\[
\sum_{i=1}^m D_i^T R_i^{-1}D_i x = \sum_{i=1}^m D_i^T R_i^{-1}y_i, \tag{28}
\]

Clearly, if \( R_i^{-1} = e_i^{-2}W_i^TW_i, \) Eq. (17) results.

We consider now the case where the \( y_i \) images are independently restored and then appropriately combined to give the final result. This independent restoration can take place in parallel or sequentially. This case can be thought of as a distributed estimation case, where each restoration filter outputs a result (probably at a remote site) and the restored images are then sent to a fusion center.

More specifically we apply the regression model of Eq. (27) to each of the observations \( y_i \) in Eq. (1). This results in

\[
\hat{x}_i = \Sigma_i D_i^T R_i^{-1}y_i, \tag{29}
\]

where

\[
\Sigma_i = E[(x - \hat{x}_i)(x - \hat{x}_i')^T] = [D_i^T R_i^{-1}D_i']^{-1} \tag{30}
\]

is the covariance matrix of the estimation error. In other words, we can now write

\[
e_i = x - \hat{x}_i \quad \text{or} \quad \hat{x}_i = x - e_i, \quad i = 1, \ldots, m. \tag{31}
\]
According to Eq. (31) the restored images' \( \hat{x}_i \)'s are now the new "observations" and \( e_i \) represents the additive noise, with its covariance matrix described by Eq. (30). Following again the same approach that was followed above we can write

\[
\hat{x} = Ix + e, \tag{32}
\]

where the vectors \( \hat{x} \) and \( e \) are of size \( mN \times 1 \) and the matrix \( I \) is of size \( mN \times N \). Since each estimate \( \hat{x}_i \) is unbiased, that is, \( E[\hat{x}_i] = x_i \), it follows that the error processes \( e_i \) are uncorrelated, that is, \( E[e_i e_j^\top] = 0 \), for \( i \neq j \). Then it holds that \( E[ee^\top] = \Sigma \), where \( \Sigma \) is a block diagonal matrix with each block equal to \( \Sigma_i \). Given Eq. (32) a regression approach results in

\[
x = [\Sigma_1^{-1} + \cdots + \Sigma_m^{-1}]^{-1}[\Sigma_1^{-1} \hat{x}_1 + \cdots + \Sigma_m^{-1} \hat{x}_m]. \tag{33}
\]

It is straightforward to show that by substituting the values for each \( \Sigma_i \) from Eq. (30) into Eq. (33), Eq. (27) is obtained. That is, the result of the simultaneous or global restoration of Eq. (27) equals the result of the independent restoration, if these restorations are appropriately combined according to Eq. (33).

5. ITERATIVE IMPLEMENTATION

Due to the sizes of the matrices in Eq. (13), it is preferable, from an implementation point of view, to use iterative algorithms in obtaining an estimate of \( x \). Furthermore, iterative restoration algorithms are popular due to other advantages they offer over noniterative algorithms [16, 17, 23]. Among these advantages is the possibility of incorporating a priori information about the solution into the iterative process in the form of constraints [23].

**FIG. 2.** Noisy–blurred images: (a) \( T_1 = 32 \) pixels, SNR = 20 dB; (b) \( T_2 = 9 \) pixels, SNR = 10 dB; (c) \( T_3 = 7 \) pixels, SNR = 3 dB.
FIG. 3. Iterative inverse filter restorations: (a) restoration of Fig. 2a, $\Delta_{\text{SNR}} = 6.58$ dB; (b) restoration of Fig. 2b, $\Delta_{\text{SNR}} = -5.16$ dB; (c) restoration of Fig. 2b, $C = 2D$ Laplacian, $\Delta_{\text{SNR}} = -0.40$ dB.

The successive approximations iteration based on Eq. (13) becomes

$$x_{k+1} = P \left[ (I - \beta E^{-2} C^T F^T F C) x_k \right. + \beta \sum_{i=1}^{m} \epsilon_i^{-2} D_i^T W_i W_i^T [y_i - D_i x_k] \left. \right]_\mu.$$  \hspace{1cm} (34)

The operator $P$ projects the signal onto a convex set which is formed by signals with a certain property, such as the positivity property [29]. Successive approximation-based iterations have been extensively studied and used [23]. Since $P$ is a nonexpansive mapping, iteration (34) converges to a unique fixed point if $0 < \beta < 2 \|E^{-2} C^T F^T F C + \sum_{i=1}^{m} \epsilon_i^{-2} D_i^T W_i W_i^T D_i\|^{-2}$ [20]. This sufficient condition for convergence simplifies to $0 < \beta < 2(E^{-2} + \sum_{i=1}^{m} \epsilon_i^{-2})^{-1}$, for $F = W = I$, since the $C$ and $D_i$ operators are normalized, which means that $\|C\|_2^2 = \|D_i\|_2^2 = 1$.

6. EXPERIMENTAL RESULTS

Experimental results obtained with the use of the pointwise version of the iterative algorithm (34) are presented in this section. The criterion $\|x_{k+1} - x_k\|^2/\|x_k\|^2 \leq 10^{-7}$ was used in our experiments for terminating the iteration. Since the original undistorted image is available in a simulation experiment, the performance of the restoration iteration is evaluated by measuring the improvement in signal-to-noise ratio, $\Delta_{\text{SNR}}$, after $k$ iterations. It is defined by

$$\Delta_{\text{SNR}} = 10 \log_{10} \left( \frac{1/m}{\sum_{i=1}^{m} \|y_i - x\|^2_2} \right) \left( \frac{\|x_k - x\|^2_2}{\|x_k - x\|^2_2} \right).$$ \hspace{1cm} (35)
In our experiments presented here, the distortion is due to the translation of the object at constant velocity along a straight line during the exposure interval. The impulse response of the distorting system is equal to [2]

\[
d_i(n) = \begin{cases} 
1/(T_i), & n = 0, 1, \ldots, T_i - 1 \\
0, & \text{otherwise,}
\end{cases} \tag{36}
\]

where \( T_i \) is the extent of the motion. Noisy and blurred images are shown in Fig. 2a \((T_i = 32, \text{blurred signal-to-noise ratio (SNR) = 20 dB})\), Fig. 2b \((T_i = 9, \text{SNR} = 10 \text{ dB})\), and Fig. 2c \((T_i = 7, \text{SNR} = 3 \text{ dB})\). A 2D Laplacian filter is used for \( C \). Inverse filter restorations of Figs. 2a and 2b, implemented by iteration (34), are shown respectively in Figs. 3a \((\Delta_{\text{SNR}} = 6.58 \text{ dB})\) and 3b \((\Delta_{\text{SNR}} = -5.16 \text{ dB})\). Ringing artifacts dominate the restoration of Fig. 3a due to the severity of the distortion, while the image in Fig. 3b is completely buried in noise. A restoration of Fig. 2b with the use of \( C \) is shown in Fig. 3c \((\Delta_{\text{SNR}} = -0.40 \text{ dB})\).

A restored image obtained by combining all three distorted images according to Eq. (13) with \( F = W = I \) and \( E = \infty \) is shown in Fig. 4 \((\Delta_{\text{SNR}} = 7.95 \text{ dB})\). Restored images obtained by combining the images of Figs. 2a and 2b are shown in Fig. 5a \((F = W = I, E = \infty, \Delta_{\text{SNR}} = 6.10 \text{ dB})\), Fig. 5b \((F = W = I, C = 2 \text{D Laplacian}, \Delta_{\text{SNR}} = 7.33 \text{ dB})\), and Fig. 5c \((F, W \text{ computed according to (24), } C = 2 \text{D Laplacian}, \Delta_{\text{SNR}} = 7.50 \text{ dB})\). The value of \( E \) is chosen according to the relation \( \|Cx\|^2 \leq \|Cy\|^2 \leq \|y\|^2 = E^2 \), since \( ||C||^2 = 1 \). The distorted image \( y_i \) with the smallest norm should be chosen in determining \( E \), provided that the resulting \( Q_i \) intersects all other ellipsoids. On the basis of the values of \( \Delta_{\text{SNR}} \) and the visual quality of the images in Figs. 4 and 5a–5c, it becomes clear that improved restoration results are obtained by combining the different distorted versions of the original image, according to the algorithms proposed in this paper. Condition (21) and the intersection of the sets have been verified and the size of the intersection has been measured according to Eqs. (19) and (20). For example, the following condition numbers have been computed: \( \mathcal{P}(D_1 D_2) = 0.75 \times 10^5 \), with 7936 singular values equal to zero; \( \mathcal{P}(D_2 D_3) = 0.22 \times 10^6 \); \( \mathcal{P}(D_1 D_3) = 0.19 \times 10^6 \); \( \mathcal{P}(\sum_{i=1}^{3} \epsilon_i^{-2} D_i D_i) = 0.28 \times 10^5 \); \( \mathcal{P}(E^{-2}C^i C + \sum_{i=1}^{3} \epsilon_i^{-2} D_i D_i) = 0.11 \times 10^5 \); with no zero singular values.

Experiments with other types of images and distortions have also been performed. The effectiveness of the positivity constraint has been tested on test images. The positivity constraint has not been shown to be very powerful for images with a large mean value, such as the portrait image used in this experiment.

7. CONCLUSIONS

A set theoretic approach has been followed in this work in deriving multiple input restoration algorithms.

These algorithms allow for the incorporation of prior knowledge about the solution into the restoration process. Furthermore, they adapt to the spatial detail of the image. The various distorted versions of the original image are combined with weights which are inversely proportional to the amount of noise. Information that has been lost in one of the noisy–blurred images is provided in the restoration process by another distorted image. An advantage of the proposed algorithms, which has been verified experimentally, is the increased tolerance of modeling errors.

The approaches presented in this work can be applied to the restoration of sequences of images, if the motion in the sequence is negligible, or if perfect motion compensation can be achieved. Clearly, if the assumption of perfect motion compensation does not hold, then Eq. (1) with \( x \) replaced by \( x_i \) represents the degradation model of multichannel images, as is commonly referred to in the literature. Hunt and Kubler [9] formulated a minimum MSE multichannel restoration filter using cross-channel correlation in the process. Galatsanos and Chin [5] derived a computationally efficient algorithm without the separability assumption. A set theoretic approach to solving the multichannel restoration problem is currently under investigation.

APPENDIX I

In this appendix the derivation of Eq. (8) from Eq. (7) is shown. Clearly, the actual observation \( y_i \) specifies a set \( Q_{\alpha y_i} \), which must contain \( x \); i.e.,

\[
x \in Q_{\alpha y_i} = \{x : (D_i x - y_i) \in Q_{\alpha_i}\}. \tag{11}
\]
FIG 5. Restored images by the proposed algorithm; the images in Figs. 2a and 2b were used: (a) $F = W = I$, $E = \infty$, $\Delta_{SNR} = 6.10$ dB; (b) $F = W = I$, $C = 2D$ Laplacian, $\Delta_{SNR} = 7.33$ dB; (c) adaptively restored image, $\Delta_{SNR} = 7.50$ dB.

More specifically, the set $Q_{xy}$ can be further written as

$$Q_{xy} = \{ x : (D_j x - y_j)^T W^T W (D_j x - y_j) \leq \epsilon_j^2 \}$$  \hspace{1cm} (12)

$$= \{ x : (D_j x - D_j x_j^*)^T W^T W (D_j x - D_j x_j^*) \leq \epsilon_j^2 \}$$  \hspace{1cm} (13)

$$= \left\{ x : (x - x_j^*) \frac{D_j^T W^T W D_j}{\epsilon_j^2} (x - x_j^*) \leq 1 \right\}$$  \hspace{1cm} (14)

In obtaining Eq. (14) it was assumed that $y_j \in \mathcal{R}(D_j)$, where $\mathcal{R}(D_j)$ denotes the range of $D_j$. Then, in this case $x_j^* = D_j^* y_j$, where $D_j^*$ denotes the generalized inverse of $D_j$.

**APPENDIX II**

Consider the general case of $M$ ellipsoids

$$Q_m = \{ x : (x - c_m)^T \Gamma_m^{-1} (x - c_m) \}, \quad m = 1, \ldots, M,$$

(11) for any $\gamma_m > 0$ such that $\sum_{m=1}^M \gamma_m \leq 1$.

The ellipsoid

$$Q_b = \{ x : (x - c_b)^T \Gamma_b^{-1} (x - c_b) \leq 1 \}$$  \hspace{1cm} (II2)

which bounds the intersection of the $M$ ellipsoids is characterized by [24]

$$c_b = \bar{\Gamma}_b \left( \sum_{m=1}^M \gamma_m \Gamma_m^{-1} c_m \right)$$  \hspace{1cm} (II3)

$$\Gamma_b^{-1} = (1 - \delta^2)^{-1} \bar{\Gamma}_b^{-1}$$  \hspace{1cm} (II4)

$$\bar{\Gamma}_b^{-1} = \sum_{m=1}^M \gamma_m \Gamma_m^{-1}$$  \hspace{1cm} (II5)

$$\delta^2 = \sum_{m=1}^M \gamma_m c_m^T \Gamma_m^{-1} c_m - c_b^T \bar{\Gamma}_b^{-1} c_b$$  \hspace{1cm} (II6)
Substituting $\Gamma^{-1} = C^T F^T F C / E^2$, $c_1 = 0$, $\Gamma^{-1} = D_1^TW D_1 / e^2$, and $c_i = x_i = (D_1^T D_1)^{-1} D_1 y_i$, for $i = 1, \ldots, m$, in the equation for the center of the bounding ellipsoid $c$, Eq. (12) is obtained.

APPENDIX III

In this appendix we derive Eq. (20) on the basis of Eqs. (18) and (19). We first show that

$$\Psi(x_c) = \sum_{i=1}^{m} \alpha_i \|y_i - y_i(x_c)\|_W^2 \geq 0 \quad \text{(II1)}$$

Consider the functional

$$\Phi(x) = \sum_{i=1}^{m} \alpha_i \|D_i x - y_i\|_W^2 + \alpha_{m+1} \|C x\|_p^2 \quad \text{(II2)}$$

This functional is minimized by $x_c$ of Eq. (12). Then, it is easily verified that the minimum value of the function $\Phi(x_c)$ is equal to $\Psi(x_c)$. Therefore, since $\delta_i(\varepsilon_i, E, y_i, x_c, \rho) \geq 0$ (prior knowledge is consistent), it follows that $\delta_i(\varepsilon_i, E, y_i, x_c, \rho)^{-\frac{1}{2}} \leq \varepsilon_i / \sqrt{\rho_i}$, which results in Eq. (20).

REFERENCES


