V. A PRELIMINARY LISTENING EXPERIMENT

Tests in [2] demonstrate the effectiveness of timbral interpolation. Other testing needs to be performed to prove the fidelity of our partial pruning algorithms. We plan a same/difference test between two renditions of symphonic passages, one rendition with all partials present and one with some number of pruned partials. So far we have performed only an informal preliminary experiment, but this preliminary experiment shows an interesting result.

Fig. 2 was produced from 600 listening trials (60 trials for each of 10 listeners). Each trial presented the listener with 2 to 5 tones played individually, and then the same tones played simultaneously in a chord. The individual tones were played with all partials present, but the chord had a variable number of partials pruned out.

The listener was asked if the chord sounded "complete." A variety of timbres were used (bowed violin, plucked violin, flute, clarinet, trumpet, muted trombone, vibraphone), as well as a variety of pitches, intervals, and durations. The horizontal axis in Fig. 2 is the sum of the number of partials in the timbre definitions for each note in the chord; the vertical axis is the number of partials which were sufficient to produce a chord synthesis which sounded "complete" to the listeners. Fig. 2 shows an important effect: the number of partials required in the chord synthesis is a maximum; in this experiment no more than 75 partials are needed. We expect that a same/difference test will also show that the number of significant partials has a maximum, and that any size ensemble may be synthesized with a fixed number of partial generators.

VI. CONCLUSION

We have presented a method for real-time Fourier synthesis of ensembles which takes advantage of masking effects and supports interpolation. In our approach, a fixed number of partial generators is used. If a few notes are being played partial generators are allocated to all of the partials for the note, and a complete Fourier synthesis is formed. A simplified Fourier synthesis is used if many notes are being played: an algorithm based on masking effects is used to determine which partials are the least important and need not be synthesized. Thus, the total computation required to synthesize any size ensemble is kept constant.

We presented an interpolation algorithm which uses a multidimensional timbre space. A 3-dimensional timbre space has been implemented on inexpensive hardware: real-time performance is made computationally feasible by precomputing and indexing frequency envelopes for timbres on regularly spaced grid points in the timbre space. Some important areas for further research include using interpolation for transitions between successive notes, improved algorithms for selecting the psychoacoustically least important partials, and listening tests to verify the fidelity of our techniques.

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Simultaneous Iterative Image Restoration and Evaluation of the Regularization Parameter

M. G. Kang and A. K. Katsaggelos

Abstract—In this correspondence, a nonlinear regularized iterative image restoration algorithm is proposed, according to which only the noise variance is assumed to be known in advance. The algorithm results from a set theoretic regularization approach, where a bound of

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the stabilizing functional, and therefore the regularization parameter, are updated at each iteration step. Sufficient conditions for the convergence of the algorithm are derived and experimental results are shown.

I. INTRODUCTION

In many practical situations, the image degradation process can be adequately modeled by a linear blur and an additive white Gaussian noise process [1]. Then the degradation model is described by

$$y = Dx + n$$

where the vectors $y$, $x$, and $n$ represent, respectively, the lexicographically ordered noisy blurred image, the deterministic original image, and the additive noise. The matrix $D$ represents the linear invariant distortion. The image restoration problem calls for obtaining an estimate of $x$ given $y$, $D$, and knowledge about the noise process.

There are a number of approaches providing a solution to the restoration problem [1]. Among these is a set theoretic regularization approach [2], [3], according to which the prior knowledge constrains the solution to belong to both ellipsoids

$$Q_1 = \{ x | \|C x\| \leq E_1^2 \}$$

and

$$Q_{1/2} = \{ x | \|y - Dx\| \leq \epsilon^2 \}$$

where $C$ represents in general a high-pass filter, so that the energy of the restored signal at high frequencies, due primarily to the amplified broad-band noise, is bounded. If the bounds $\epsilon^2$ and $E_1^2$ are known, and the intersection of $Q_1$ and $Q_{1/2}$ is not empty, an $x$ belonging to this intersection satisfies

$$D^T D + \alpha C^T C x = D^T y$$

where $\alpha$, the regularization parameter, is equal to $(\epsilon / E_1)^2$ and $C$ denotes the transpose of a vector or a matrix. The solution of (4) represents the center of one of the ellipsoids that include the intersection of the two ellipsoids [3]. It also represents the solution obtained by Miller’s regularization approach [2].

The regularization parameter controls the tradeoff between fidelity to the data and smoothness of the solution, and therefore its determination is a very important issue. If only one of the bounds $\epsilon$ or $E_1$ is known, a constrained least squares (CLS) approach [4] can be followed. According to such an approach, if only $\epsilon$ is known, the size of $Q_1$ is minimized subject to the constraint that the solution is on the surface of $Q_{1/2}$; if only $E_1$ is known, the dual problem is solved. In this case $\alpha$ represents a Lagrange multiplier which needs to be specified iteratively, resulting in an additional computational overhead. Other approaches for determining the regularization parameter are based, for example, on the methods of cross validation [5] and maximum likelihood [6]. For these cases, knowledge of the noise variance is required, and the weighted least squares criterion (minimization of $E[\|D x - D y\|^2]$), where $E$ denotes expectation and $x$ is the solution satisfying (4)) and the sum of squared weighted residuals criterion (minimization of $E[\|A(\sigma^2)^{1/2} (A x - y)\|^2]$), where $A = D D^T + \alpha^{-1} C^T C$ $D^T y$ require knowledge of the noise variance [7]. With all these methods the determination of the regularization parameter is a separate first step followed by the restoration of the image.

An alternative approach for the determination of the regularization parameter is presented in this correspondence, based on the formulation of (2) and (3). Clearly, knowledge of $E_1^2$ and $\epsilon^2$ results in the direct determination of $\alpha$, therefore reducing the computational load. Various approaches exist for estimating the noise variance (and therefore $\epsilon^2$). Such an estimate can be obtained, for example, by considering the power spectrum of a flat region of the image [1], and recursively [8] or iteratively [9] based on the maximum likelihood principle. However, no specific suggestions have been made for obtaining a tight bound on $\|C x\|$ in our previous work [2], [3] loose bounds on $\|C x\|$ were used, resulting in rather noisy restorations. We propose an iterative restoration algorithm according to which $E$ is updated at each iteration step based on the available restored image. At convergence a tight bound on $\|C x\|$ is obtained. The resulting nonlinear iteration is presented in Section II and its convergence analysis in Section III. Experimental results are shown and discussed in Section IV, and Section V concludes the correspondence.

II. AN ITERATIVE RESTORATION ALGORITHM

Iterative algorithms offer a number of advantages in solving restoration problems [10]. The successive approximation of the solution of (4) results in the iterative algorithm [2], [3]

$$x_{k+1} = I - \beta \alpha C^T C x_k + \beta D^T (y - D x_k)$$

where $\beta > 0$ is chosen to insure convergence and maximize the rate of convergence. Iteration (5) has been analyzed in detail in [3], when $\alpha$, or equivalently both bounds $E_1$ and $\epsilon^2$ in (2) and (3), respectively, are known, and also when hard constraints are used. From these two words, $\epsilon^2$ is equal to $N \sigma^2$, where $N \times N$ is the size of all images involved and $\sigma^2$ is the variance of the zero-mean noise. There are a number of different approaches to obtain an accurate estimate of $\sigma^2$ (see, for example [1], [6]--[8]). No specific procedures have been given in the literature in determining a tight bound of $\|C x\|$ in (2). We propose to use the available estimate of the restored image at each iteration step to obtain an estimate of $E$. In other words, we define $E_k = \|C x_k\| + \delta$, and therefore propose the following iteration:

$$x_0 = \beta D^T y$$

$$x_{k+1} = \left( I - \beta \frac{\epsilon^2}{\|C x_k\|^2 + \delta} \right) x_k + \beta D^T (y - D x_k)$$

where $\beta > 0$ again has to be chosen to insure convergence of the iteration and $\delta$ is a compensation factor to keep the bound $\|C x\|$ away from zero. A lower bound for $\delta$ will be determined in Section III. According to this iteration, only knowledge of the noise variance is assumed, as is the case with the CLS approach. Iteration (6), however, is computationally more efficient than the CLS approach, since estimates of the original image and the regularization parameter are obtained simultaneously.

III. CONVERGENCE ANALYSIS

Sufficient conditions for the convergence of iteration (6) are derived in this section. It can be written as

$$x_{k+1} = x_k + \beta (D^T y - D^T D x_k - \epsilon^2 f(x_k))$$

where the nonlinear factor $f(x_k)$ is equal to

$$f(x_k) = \frac{C^T C x_k}{C^T C x_k + \delta}$$

Rewriting iteration (7) for two consecutive values of $k$ we obtain

$$x_{k+1} - x_k = (I - \beta \frac{\epsilon^2}{\|C x_k\|^2 + \delta}) f(x_k) - f(x_{k-1})$$


According to the mean value theorem

\[ x_{k+1} - x_k = [I - \beta(D^TD + \epsilon^2 J_f(\xi))](x_k - x_{k-1}) \]

(10)

where \( J_f(\xi) \) is the Jacobian matrix of \( f \) at \( \xi \) (a vector between \( x_k \) and \( x_{k-1} \)). Let us set \( \xi = Cr \). Then since \( J_f(x) = J_f(c) \cdot J_a(x) \) [11], we obtain

\[
J_f(x) = \frac{C^T}{(\|z\|^2 + \delta)^2} \begin{bmatrix}
    \|z\|^2 + \delta & -2z_1z_2 & \cdots & -2z_1z_n \\
    -2z_2z_1 & \|z\|^2 + \delta & \cdots & \cdots \\
      \cdots & \cdots & \cdots & \cdots \\
    -2z_nz_1 & \cdots & \cdots & \|z\|^2 + \delta - 2z_nz_n
\end{bmatrix} C
\]

(11)

\[
= \frac{C^T C}{\|z\|^2 + \delta} - \frac{2}{\|z\|^2 + \delta} \frac{C^T C}{(\|z\|^2 + \delta)^2} C
\]

(12)

\[
= C^T (J_a - J_a^f) C
\]

(13)

where \( z_i \) is the \( i \)th component of the vector \( z \), and \( J_a \) is a diagonal matrix with constant elements equal to \( 1/(\|z\|^2 + \delta) \). The second term in (13) is equal to

\[
C^T J_f C = \frac{2C^T C R_a C^T C}{(\|z\|^2 + \delta)^2}
\]

(14)

where \( R_a = xx^T \) is the autocorrelation matrix of the deterministic signal \( x \). Then according to (13) and (14), (10) can be written as

\[
x_{k+1} - x_k = \left[ I - \beta \left( D^TD + \epsilon^2 \frac{C^T C}{(\|z\|^2 + \delta)^2} \right) \right] (x_k - x_{k-1})
\]

where \( 0(h^2) \) represents the higher order terms. In writing (15) we have assumed that the term \( (x_k - x_{k-1}) \) is very small (first-order zero function), therefore the term \((1/2)(x_k - x_{k-1})^T \Theta (x_k - x_{k-1})\), where \( F \) denotes the Hessian of \( f \) and \( 0 \approx \theta \leq 1 \), has been omitted. We assume next that \( R_a \) is approximated by a block Toeplitz matrix. This assumption is commonly made and is justified if \( x_k \), which represents a deterministic signal in our formulation, is thought of as a realization of a wide-sense stationary and ergodic random field. We further approximate \( R_a \) as well as \( D^TD \) and \( C^TC \), which are also block Toeplitz matrices, by block circulant matrices. Then (15) is written in the discrete frequency domain as

\[
X_{k+1}(l) - X_k(l) = \left[ 1 - \beta \left( \frac{D^TD}{\|z\|^2 + \delta} + \frac{\epsilon^2 |C(\xi)|^2}{(\|z\|^2 + \delta)^2} \right) \right] (X_k(l) - X_{k-1}(l))
\]

(16)

where \( l = (l_1, l_2) \), \( m = (m_1, m_2) \), with \( 0 \leq l_1 \leq N - 1, 0 \leq m_1 \leq N - 1, 0 \leq l_2 \leq M - 1, 0 \leq m_2 \leq M - 1 \), \( X(l) \) represents the two-dimensional (2D) discrete Fourier transform (DFT) of the unstacked image \( x \), and \( D(l) \) and \( C(l) \) represent the 2D DFT's of the 2D sequences which form the block-circulant matrices \( D^TD \) and \( C^TC \), respectively. If we consider the magnitude of both sides of (16), we obtain with the use of the triangular inequality

\[
|X_{k+1}(l) - X_k(l)| \leq \left| 1 - \beta \left( \frac{D(l)}{\|z\|^2 + \delta} + \frac{\epsilon^2 |C(l)|^2}{(\|z\|^2 + \delta)^2} \right) \right| \cdot |X_k(l) - X_{k-1}(l)|
\]

(17)

According to (17), the condition

\[
1 - \beta \left( \frac{D(l)}{\|z\|^2 + \delta} + \frac{\epsilon^2 |C(l)|^2}{(\|z\|^2 + \delta)^2} \right) < 1
\]

(18)

is sufficient for the convergence of the iteration. In order for inequality (18) to be satisfied,

\[
H_c(l) = \frac{D(l)^2 + \epsilon^2 |C(l)|^2}{\|z\|^2 + \delta} - \frac{2\epsilon^2 |C(l)|^4 |X(l)|^2}{\sum_m (|C(m)X(l)|^2 + \delta)} < 0
\]

(19)

should be strictly positive and \( \beta \) should satisfy

\[
0 < \beta < M_k
\]

(19)
TABLE I

| SNR (dB) | \(\sigma^2\) | \(M\) | \(\beta\) | \(\delta\) | \(l\) | \(|C(l)|^2\) | \(|D(l)|^2\) | \(\Delta_{\text{SNR}}\) (dB) |
|----------|-------------|-----|-----|-----|-----|------|------|-------|
| 10       | 216.1       | 0.60665 | 0.54598 | 3.737696.75 | 90, 44 | \(1.7313 \times 10^{-2}\) | \(3.7337 \times 10^{-2}\) | 6.01 (\(k = 78\)) |
| 20       | 216.1       | 2.00000 | 1.80000 | 3.737742.75 | 54, 21 | \(1.1477 \times 10^{-2}\) | \(2.4943 \times 10^{-3}\) | 7.72 (\(k = 73\)) |
| 30       | 2.161       | 2.00000 | 1.80000 | 3.736885.75 | 16, 80 | \(1.2725 \times 10^{-1}\) | \(2.5750 \times 10^{-1}\) | 10.14 (\(k = 81\)) |

\[ \mathbf{B} = 6.4225 \times 10^6, \quad \mathbf{\varepsilon} = 65536 \times \sigma^2. \]

Fig. 1. Values of \(\alpha_k\) for various SNR’s.

Fig. 2. Values of \(|y - D_{a_k}|^2/\mathbf{N}^2\) for various SNR’s.

where

\[
M_k = \frac{2}{\max_i \left[ |D(l)|^2 + \frac{\varepsilon^2|C(l)|^2}{\|C(m)X(m)\|^2} + \delta \right]}. \tag{20}
\]

According to (19), \(\beta\) depends on the iteration index, since \(M_k\) does not.

However, a lower bound for \(M_k\) independent of \(k\) is given by

\[
M = \frac{2}{\max_i \left[ |D(l)|^2 + \frac{\varepsilon^2|C(l)|^2}{\|C(m)X(m)\|^2} + \delta \right]} \tag{21}
\]

\(M\) is strictly positive, since \(0 \leq \|D(l)| \leq 1\) and \(0 \leq \|C(l)| \leq 1\), and therefore, a \(\beta\) satisfying \(0 < \beta < 3\) can always be found.

The condition \(H_k(l) > 0\) is used in establishing a bound on \(\delta\); it can be rewritten as

\[
|D(l)|^2 \delta^2 + \left[ 2|D(l)|^2 \sum_m |C(m)X(m)|^2 + \varepsilon^2|C(l)|^2 \right] \delta \nonumber \]

\[
+ \left[ |D(l)|^2 \left( \sum_m |C(m)X(m)|^2 \right)^2 \right] \nonumber \]

\[
+ \varepsilon^2 |C(l)|^2 \left( \sum_m |C(m)X(m)|^2 - 2\varepsilon^2|C(l)|^2|C(l)X(l)|^2 \right) \geq 0. \tag{22}
\]

If we assume that for some \(B\),

\[
|X_k(l)| \leq B, \quad \forall k, l \tag{23}
\]

then (22) is satisfied for

\[
\delta \geq \max_l \left[ -\varepsilon^2|C(l)|^2 + |C(l)|^2\sqrt{\varepsilon^2 + 8\varepsilon^2|D(l)|^2\delta^2} \right] \nonumber \]

\[
2|D(l)|^2 \nonumber \]

\[
\text{for } |D(l)| \neq 0, \tag{24}
\]

\[
\delta \geq 2B \quad \text{for } |D(l)| = 0. \tag{25}
\]

A sufficient condition for (23) to hold true is that all entries of the vector \(x_k\) are nonnegative for all \(k\). Then, let \(m_k\) and \(m_\beta\) denote the mean values of the original and distorted images, respectively. It can be easily verified that \(m_k = m_\beta\), and that iteration (6) preserves the mean value of the image, for \(0 < \beta < 2\), since the noise has zero mean, \(D(0) = 1\) (it is assumed that the degradation system preserves energy) and \(C(0) = 0\), since \(C\) represents a high-pass filter (the 2D Laplacian is used in our experiments). Therefore, \(X_k(l) \leq \sum_{n \in \mathbb{N}} |x_k(l)| = N^2 \times m_\beta = N^2 \times m_\beta = B\). This assumption includes a large number of grey-scale images of interest, as has been shown to hold true for a large number of experiments.

If assumption (23) does not hold true for all \(k\), then the above presentation becomes a heuristic argument which suggests that the iteration converges, as has been verified in all our experiments. Summarizing, iteration (6) is guaranteed to converge if \(\beta\) satisfies \(0 < \beta < 3\), where \(M\) is given by (21), and (23)–(25) are satisfied.

IV. EXPERIMENTAL RESULTS

A number of experiments have been performed with the proposed algorithm. Some of these results are presented in this section, where a \(256 \times 256\) pixels portrait image was used, the 2-D Laplacian was used for \(C\), the blur was due to motion over 9 pixels, and the criterion \(\|x_k - x_l\|/\|x_k\| \leq 10^{-6}\) was used for terminating the iteration. The performance of the restoration algorithm was evaluated by measuring the improvement in signal to noise ratio after \(k\)-iterations denoted by \(\Delta_{\text{SNR}}\) and defined by

\[
\Delta_{\text{SNR}} = 10 \log_{10} \left( \frac{\|y - x\|^2}{\|x_k - x_l\|} \right). \tag{26}
\]

The values of \(\Delta_{\text{SNR}}\) and the required number of iterations, as well as the values of the parameters \(\sigma^2, M, \beta,\) and \(\delta\) are shown in Table I for signal-to-noise ratios (SNR’s) of 10, 20, and 30 dB. Also shown in this table are the discrete frequency locations \(l\) where
value of $\delta$ is determined according to (24), (25), and the corresponding values of $|C(I)|^2$ and $|D(I)|^2$. The bound $B$ in (23) is equal to $N^2 \cdot m_2 = 6.4225 \times 10^6$, where $N = 256$ and $m_2 = m_3 = 98$. The values of the regularization parameter $\alpha$, the residual $\|y - D x\|^2$ and the error between iteration steps $\|x_k - x_{k-1}\|^2$, for the three SNR's mentioned above are compared in Figs. 1–3, respectively. Notice that the plots are scaled appropriately and that the vertical axis in Fig. 3 is logarithmic. It is observed in Fig. 1 that in all cases $\alpha$ converges to a value close to $(N^2 \sigma_x^2)/\delta$. This is expected, since the better the whitening of $\gamma$ by $C$, the closer $\alpha$ is to $(N^2 \sigma_x^2)/\delta$. For $C^T C = R_x^{-1}$, it can be shown [6], [7] by using a variety of methods for determining the regularization parameter, that $\alpha = \varepsilon^2$ if there is no $\delta$. As shown in Fig. 2, the residuals converge to a value smaller than $\sigma_x$. This result is also expected, since the solution at convergence (center of an ellipsoid bounding the intersection of $Q_x$ and $Q_{x/\gamma}$ in (2) and (3), respectively) lies inside the ellipsoid $Q_{x/\gamma}$. The plots in Fig. 3 demonstrate that in all cases the iterative algorithm converges with the same rate. The noisy blurred image for SNR = 10 dB is shown in Fig. 4(a), and three restored images are shown in Figs. 4(b) (SNR = 10 dB, ...
TABLE II
COMPARISON OF REGULARIZATION PARAMETERS

| SNR (dB) | min. \( |\varepsilon| - \alpha_{\text{org}}|^2 \) | \( \frac{\alpha_{\text{opt}}}{e^2/(|C_{\text{org}}|^2)} \) | \( \frac{\alpha_{\text{opt}, 0}}{e^2/(|C_{\text{org}}|^2)} \) | \( \frac{\alpha_{\text{opt}}}{e^2/(|C_{\text{org}}|^2) + \delta} \) | min. CV function |
|---------|---------------------------------------------|---------------------------------------------|---------------------------------------------|---------------------------------------------|----------------|
| 10      | 15.9                                       | 15.8                                       | 7.03                                        | 3.44                                        | 0.571          |
| 20      | 1.35                                        | 1.09                                       | 0.703                                       | 0.310                                       | 0.0128         |
| 30      | 0.120                                       | 3.0804                                    | 0.0703                                       | 0.0280                                       | 0.00112         |

\( \Delta_{\text{SNR}} = 6.01 \text{ dB}, \Delta_{\text{SNR}} = 20 \text{ dB}, \Delta_{\text{SNR}} = 7.72 \text{ dB} \) and 4(d) (SNR = 30 dB, \( \Delta_{\text{SNR}} = 10.14 \text{ dB} \). In all cases the restored images are very satisfactory, based on the improvement in SNR and visual inspection. We finally compare the values of \( \alpha \) obtained by the proposed approach at convergence (denoted by \( \alpha_{\text{opt}} = (e^2/(|C_{\text{org}}|^2) + \delta \), where \( \alpha \) is the restored image at convergence) with the values obtained by the cross-validation method (\( \alpha_{\text{CV}} \)) and from the minimization of \( \ell(\alpha) - \alpha_{\text{org}}^2 \), denoted by \( \alpha_{\text{opt}} (\alpha_{\text{org}} \) represents the original image which is available in a simulation experiment). We further compare these values of \( \alpha_{\text{opt}, 0} = (e^2/(|C_{\text{org}}|^2)) \) and \( \alpha_{\text{opt}, 0} = (e^2/(|C_{\text{org}}|^2)) \), that is the value of \( \alpha \) obtained by using the restored image obtained by iteration (6) but setting \( \delta = 0 \). All these values are shown in Table II for three different SNR's. From this table it is clear that \( \alpha_{\text{opt}} > \alpha_{\text{opt}, 0} > \alpha_{\text{opt}, 0} > \alpha_{\text{opt}} > \alpha_{\text{CV}} \). This implies that the restored images obtained with the use of these values of the regularization parameter will be increasingly smoother as \( \alpha \) increases from \( \alpha_{\text{opt}} \) to \( \alpha_{\text{opt}} \). We also observe that \( \alpha_{\text{opt}, 0} = \alpha_{\text{opt}} \) making the restored image obtained with the use of \( \alpha_{\text{opt}, 0} \) optimal in the sense that \( \ell(\alpha) - \alpha_{\text{org}}^2 \) is minimized. At the same time, as discussed in [6] and the references therein, the restored image with the use of \( \alpha_{\text{opt}} \) is oversmooth, making the image obtained with the use of \( \alpha_{\text{opt}} \) subjectively more preferable.

V. CONCLUSION

In this correspondence we have proposed a regularized iterative image restoration algorithm according to which a restored image and an estimate of the regularization parameter are provided simultaneously at each iteration step. Sufficient conditions for the convergence of the algorithm have been derived. The algorithm assumes the same amount of prior knowledge as the CLS algorithm, namely, knowledge of the value of the variance of the noise, but requires considerably fewer computations. Linear constraints can also be incorporated into the iteration. When nonlinear constraints are used the linearization step in the analysis of the algorithm may not be applied, although the algorithm has been shown to converge experimentally. Although the analysis is carried out for the case that \( D \) and \( C \) represent spatially invariant filters and the autocorrelation matrix of the image is block circulant, iteration (6) can be run for the more general case when \( D \) and \( C \) represent spatially varying filters and the image is nonstationary. The convergence analysis holds true in the general case as well, although it becomes more computationally expensive to verify the sufficient conditions for convergence at each iteration step. In other words, no explicit bounds for \( \beta \) and \( \delta \) can be reached in this case.

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On the Design of FIR Digital Differentiators which Are Maximal Linear at the Frequency \( \pi/p, p \in \{\text{Positive Integers}\} \)

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Abstract—In a number of signal processing applications, a digital differentiator (DD) performing over a narrow band of frequencies is required. The minimax relative error DD's are especially suitable for broad-band frequencies and become inefficient when adapted for narrow-band signals. The maximally linear DD's are, therefore, preferred for the latter situations. This correspondence proposes digital differentiators which are maximally linear at the spot frequency \( \pi/p, p \in \{\text{positive integers}\} \). The suggested DD's, besides giving zero phase error over the entire band of frequencies \(-\pi \leq \omega \leq \pi\), can achieve very high accuracy in the magnitude response, over a given frequency range.

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