FREQUENCY DOMAIN ADAPTIVE ITERATIVE IMAGE RESTORATION AND EVALUATION OF THE REGULARIZATION PARAMETER

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ABSTRACT

In this paper a nonlinear frequency domain adaptive regularized iterative image restoration algorithm is proposed, according to which the regularization parameter is frequency dependent and is updated at each iteration step. The development of the algorithm is based on a set theoretic regularization approach, where bounds on the error residual and the stabilizing functional are updated in the frequency domain at each iteration step. Sufficient conditions for the convergence of the algorithm are derived and experimental results are shown.

1. INTRODUCTION

The image degradation process can be adequately modeled by a linear blur and an additive white Gaussian noise process [1, 2]. Then the degradation model is described by

\[ y = Dx + n, \]

where the vectors \( y, x \) and \( n \) represent, respectively, the lexicographically ordered noisy blurred image, the deterministic original image and the additive noise. The matrix \( D \) represents the linear spatially invariant distortion. The image restoration problem calls for obtaining an estimate of \( x \) given \( y, D \), and some knowledge about the noise process.

There are a number of approaches providing a solution to the restoration problem [1, 2]. Among these is a set theoretic regularization approach [3, 4, 5], according to which the prior knowledge constrains the solution to belong to both ellipsoids

\[ Q_x = \{ x | \|Cx\|^2 \leq E^2 \}, \]

and

\[ Q_{x/y} = \{ x | \|y - Dx\|^2 \leq \epsilon^2 \}, \]

where \( C \) represents in general a high-pass filter, so that the energy of the restored signal at high frequencies, due primarily to the amplified broad-band noise, is bounded. If the bounds \( \epsilon^2 \) and \( E^2 \) are known, and the intersection of \( Q_x \) and \( Q_{x/y} \) is not empty, an \( x \) belonging to this intersection satisfies

\[ (D^T D + \alpha C^T C)x = D^T y, \]
where \( \alpha \), the regularization parameter, is equal to \( (\frac{\epsilon}{T})^2 \) and \( T \) denotes the transpose of a vector or a matrix. The solution of Eq. (4) represents the center of one of the ellipsoids that include the intersection of the two ellipsoids \([3, 4, 5]\). It also represents the solution obtained by Miller's regularization approach \([6, 7]\). The determination of the regularization parameter is very important, since it controls the trade-off between fidelity to the data and smoothness of the solution. Various techniques exist in the literature for the evaluation of the regularization parameter, which depend on the assumed prior knowledge about the problem \([8]-[11]\). With all these methods the determination of the regularization parameter is a separate first step followed by the restoration of the image.

In this paper, we propose a frequency adaptive iterative restoration algorithm according to which both \( E \) and \( \epsilon \) are updated in the frequency domain at each iteration step based on the available restored image. This way, a different regularization parameter is determined at each discrete frequency. The case when the same regularization parameter is determined for each discrete frequency is studied in \([12]\) and \([13]\), without and with adaptivity in the iteration domain, respectively. At convergence tight bounds on \( \|Cx\| \) and \( \|y - Dx\| \) are obtained. The derivation and analysis of the regularized iterative restoration algorithm is presented in Sec. 2. Determination of the regularization parameter is shown in Sec. 3, based on the convergence analysis as well as the optimality criterion. Experimental results are shown and discussed in Sec. 4, and Sec. 5 concludes the paper.

2. THE PROPOSED FREQUENCY DOMAIN ITERATION

With the set theoretic approach \([5]\) followed in this work, the prior knowledge constrains the solution to certain sets. Therefore, consistency with all the prior knowledge pertaining to the original image serves as an estimation criterion. The basic idea behind the proposed algorithm is to utilize the information available at each iteration about the restored image in determining the bounds \( \epsilon^2 \) and \( E^2 \) which define the ellipsoids in Eqs. (2) and (3). In other words, we define the ellipsoids \( Q_x \) and \( Q_{x/y} \) by choosing for bounds

\[
E_{k,s}^2 = \|Cx_k\|^2 + \delta_{k,s},
\]

and

\[
\epsilon_k^2 = \|y - Dx_k\|^2
\]

where \( \delta_{k,s} \) is a block circulant matrix required to guarantee convergence of the resulting iteration at all frequency components as will be shown next. A solution in the intersection of the ellipsoids defined in terms of the bounds (5) and (6) is given by

\[
(D^T D + \alpha_{k,s} C^T C)x = D^T y,
\]

where \( \alpha_{k,s} \) is a regularization matrix defined as

\[
\alpha_{k,s} = \epsilon_k^2 [E_{k,s}^2]^{-1}.
\]

Since matrices \( D, C \) and \( \delta_{k,s} \) have all been assumed to be block circulant Eq. (7) can be written in the discrete frequency domain as

\[
(|D(l)|^2 + \alpha_{k}(l)|C(l)|^2)X(l) = D^*(l)Y(l),
\]

where \( l = (l_1, l_2), 0 \leq l_1 \leq N - 1, 0 \leq l_2 \leq N - 1, X(l) \) and \( Y(l) \) represent respectively the two-dimensional (2D) DFT of the unstacked vectors \( x \) and \( y \), and \( D(l), C(l) \) and \( \alpha_{k}(l) \) represent 2D DFTs...
of the 2D sequences which form the block-circulant matrices $D$, $C$ and $\alpha_k$, respectively. We propose the following frequency domain successive approximations iteration to solve Eq. (9)

$$X_0(l) = \beta(l) D^*(l) Y(l),$$

$$X_{k+1}(l) = (1 - \beta(l) \alpha_k(l) |C(l)|^2) X_k(l) + \beta(l) D^*(l) (Y(l) - D(l) X_k(l)),$$

where $\beta(l) > 0$ is chosen to insure convergence at each frequency component $l$ and $\alpha_k$ is defined by

$$\alpha_k(l) = \frac{\sum_m |Y(m) - D(m) X_k(m)|^2}{\sum_n |C(n) X_k(n)|^2 + \delta_k(l)},$$

where $\delta_k(l)$ is a control parameter required to guarantee convergence.

The nonlinear iteration (10) may have a number of fixed points. Therefore, the initial condition $X_0$ is very important. We have at least two fixed points for iteration (10). The first fixed point is the inverse or generalized inverse solution of Eq. (1). Clearly, if $D(l)$ is different than zero, $X(l) = Y(l)/D(l)$ is a fixed point of iteration (10), as is easily verified. If $D(l)$ becomes zero at certain frequencies, solutions which minimize $\sum_m |Y(m) - D(m) X_k(m)|^2$, that is, satisfy $|D(l)|^2 X(l) = D^*(l) Y(l)$, are also fixed points of iteration (10). This is the case because for those frequencies for which $D(l) \neq 0$, the inverse solution is obtained which is indeed a fixed point. For the frequencies for which $D(l) = 0$, the solution $X(l) = 0$ is obtained, which is the generalized inverse solution. The second type of fixed points are represented by images which are between the “very rough” generalized inverse solution and the “very smooth” solutions based on prior knowledge. Such solutions clearly exist and are obtained by considering various constant values of $\alpha$. Since there are more than one solutions to iteration (10), the determination of the initial condition becomes important. That is, if $X_0(l) = Y(l)/D(l)$ is used, the solution $Y(l)/D(l)$ is obtained, i.e., $\alpha_k(l) \rightarrow 0$. If on the other hand a “smooth” image is used for $X_0(l)$, then the iteration always converges to almost the same regularized solution, independently of $X_0(l)$, as was discussed in [13]. This is the reason for using $X_0(l) = D^*(l) Y(l)$ in (10). Since we have more than one fixed points, the sufficient condition for convergence of iteration (10) is derived in the next section, based on the assumption that $X_k(l)$ is in the vicinity of a fixed point.

### 3. Determination of the Regularization Parameter

In this section, a sufficient condition for the convergence of iteration (10) is derived, resulting in a lower bound for the control parameter $\delta_k(l)$. An upper bound for this parameter is also derived by considering an optimality criterion for the solution.

#### 3.1 Convergence analysis

The frequency iteration Eq. (10) can be rewritten as

$$X_{k+1}(l) = X_k(l) + \beta(l)[D^*(l) Y(l) - |D(l)|^2 X_k(l) - F(X_k(l))],$$

where the nonlinear factor $F(X_k(l))$ is equal to

$$F(X_k(l)) = \frac{\sum_m |Y(m) - D(m) X_k(m)|^2 |C(l)|^2 X_k(l)}{\sum_n |C(n) X_k(n)|^2 + \delta_k(l)}.$$

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Rewriting iteration (12) for two consecutive values of k, we obtain

\[ X_{k+1}(l) - X_k(l) = (1 - \beta(l)|D(l)|^2)(X_k(l) - X_{k-1}(l)) - \beta(l)(F(X_k(l)) - F(X_{k-1}(l))). \]  

(14)

According to the Taylor series expansion of \( F(X_k(l)) \)

\[ X_{k+1}(l) - X_k(l) = [1 - \beta(l)(|D(l)|^2 + \sum_n |Y(n) - D(n)X_k(n)|^2|C(l)|^2 \sum_m |C(m)X_k(m)|^2 + \delta_k(l)] \cdot (X_k(l) - X_{k-1}(l)) \]

\[ + O(h^2), \]

(15)

where \( O(h^2) \) is the second order zero term of the iteration step \( X_k(l) - X_{k-1}(l) \) at frequency \( l \). In writing Eq. (15) we have assumed that the term \( (X_k(l) - X_{k-1}(l)) \) is very small (first order zero function), therefore the term \( \frac{1}{2}(X_k(l) - X_{k-1}(l))^2F''(X_k(l)) \), where \( F'' \) denotes the second derivative of \( F \), has been omitted. If we consider the magnitude of both sides of Eq. (15), we obtain with the use of the triangular inequality

\[ |X_{k+1}(l) - X_k(l)| \leq |1 - \beta(l)(|D(l)|^2 + \sum_n |Y(n) - D(n)X_k(n)|^2|C(l)|^2 \sum_m |C(m)X_k(m)|^2 + \delta_k(l)) \cdot |X_k(l) - X_{k-1}(l)|. \]

(16)

According to (16) the condition,

\[ |1 - \beta(l)(|D(l)|^2 + \sum_n |Y(n) - D(n)X_k(n)|^2|C(l)|^2 \sum_m |C(m)X_k(m)|^2 + \delta_k(l)) \cdot |X_k(l) - X_{k-1}(l)| < 1, \]

(17)

is sufficient for the convergence of the iteration. In order for inequality (17) to be satisfied, \( H_k \) defined as,

\[ H_k(l) = |D(l)|^2 + \sum_n |Y(n) - D(n)X_k(n)|^2|C(l)|^2 \sum_m |C(m)X_k(m)|^2 + \delta_k(l) \]

\[ - \frac{2 \sum_n |Y(n) - D(n)X_k(n)|^2|C(l)|^2 |X_k(l)|^2}{(\sum_m |C(m)X_k(m)|^2 + \delta_k(l))^2} \cdot |X_k(l) - X_{k-1}(l)|, \]

(18)

should be strictly positive and \( \beta(l) \) should satisfy

\[ 0 < \beta(l) < M_k(l) \]

(19)

where \( M_k(l) = \frac{2}{H_k(l)} \). A lower bound for \( M_k(l) \) independent of iteration \( k \) is given by

\[ M(l) = \frac{2}{\max_l(|D(l)|^2 + \sum_n |Y(n)|^2|C(l)|^2 \delta_{\min}(l))}, \]

(20)

where \( \delta_{\min}(l) \) is the minimum value over all iterations at a frequency \( l \) and is determined experimentally. \( M(l) \) is strictly positive, since \( 0 \leq |D(l)| \leq 1 \) and \( 0 \leq |C(l)| \leq 1 \), and therefore, a \( \beta(l) \) satisfying \( 0 < \beta(l) < M(l) \) can always be found at every frequency component.
The condition $H_k(l) > 0$ is used in establishing a bound on $\delta_k(l)$; it results in
\[
\delta_k(l) \geq \frac{-2|D(l)|^2 \sum_m |C(m)X_k(m)|^2 - \sum_n |Y(n) - D(n)X_k(n)|^2 |C(l)|^2}{2|D(l)|^2} + \frac{|C(l)| \sqrt{|C(l)|^2 (\sum_n |Y(n) - D(n)X_k(n)|^2)^2 + 8|D(l)|^2 |C(l)|X_k(l)^2 \sum_n |Y(n) - D(n)X_k(n)|^2}}{2|D(l)|^2} = \delta_k^{\text{conv}}(l).
\]

### 3.2 Optimality criterion

In this section, an analysis is presented of the error between the original image and the regularized estimate, following [11]. It leads to the determination of an optimal range for the parameter $\delta_k$ in Eq. (11). Let us denote by $x^+$ the minimum norm least squares solution to Eq. (1), that is, $x^+ = (D^T D)^{-1} D^T y$. Then the regularized solution $\hat{x}(\alpha)$ can be written as
\[
\hat{x}(\alpha) = (D^T D + \alpha C^T C)^{-1} D^T y = (D^T D + \alpha C^T C)^{-1} D^T D x^+ = P(\alpha)x^+,
\]
where $P(\alpha) = (D^T D + \alpha C^T C)^{-1} D^T D$. The mean square error (MSE) is given by
\[
E[\|e(\alpha)\|^2] = E[\|x - \hat{x}(\alpha)\|^2] = E[(x^+ - x)^T P(\alpha)^T P(\alpha)(x^+ - x)] + E[(P(\alpha)x - x)^T (P(\alpha)x - x)],
\]
where $E[\cdot]$ denotes the expectation operator. Since $E[\hat{x}(\alpha)] = P(\alpha)x$, the first term of Eq. (26) is equal to the variance of $\hat{x}(\alpha)$ while the second term is equal to the bias of the estimate $\hat{x}$. Since $D$ and $C$ are block circulant matrices, the variance and bias term in Eq. (26) can be described in the discrete Fourier transform (DFT) domain as
\[
\text{Var}[\hat{x}(\alpha)] = \left[\frac{1}{N^2} \sum_{m=1}^{N^2} |Y(m) - D(m)\hat{X}(m)|^2 \right] \left[\sum_{l=1}^{N^2} \frac{|D(l)|^2}{(|D(l)|^2 + \alpha|C(l)|^2)^2} \right],
\]
\[
\text{bias}[\hat{x}(\alpha)] = \sum_{l=1}^{N^2} \frac{|\hat{X}(l)|^2 \alpha^2 |C(l)|^4}{(|D(l)|^2 + \alpha|C(l)|^2)^2},
\]
Taking their derivatives with respect to $\alpha$ yields
\[
\frac{\partial \text{Var}[\hat{x}(\alpha)]}{\partial \alpha} = -2 \left[\frac{1}{N^2} \sum_{m=1}^{N^2} |Y(m) - D(m)\hat{X}(m)|^2 \right] \left[\sum_{l=1}^{N^2} \frac{|D(l)|^2 |C(l)|^2}{(|D(l)|^2 + \alpha|C(l)|^2)^3} \right] < 0,
\]
\[
\frac{\partial \text{bias}[\hat{x}(\alpha)]}{\partial \alpha} = 2 \sum_{l=1}^{N^2} \frac{\alpha |\hat{X}(l)|^2 |C(l)|^4 |D(l)|^2}{(|D(l)|^2 + \alpha|C(l)|^2)^3} > 0.
\]
From Eqs. (27) and (29) we conclude that the variance is a strictly positive, monotonically decreasing function of $\alpha$, for $\alpha > 0$ and that $\text{Var}[\hat{x}(\infty)] = 0$. Similarly, from Eqs. (28) and (30) we conclude that
the bias is a positive, monotonically increasing function of $\alpha$, for $\alpha > 0$ and that $bias[\hat{x}(0)] = 0$ and $bias[\hat{x}(\infty)] = \|x\|^2$. Thus, the total error as a function of $\alpha$ is a sum of a monotonically decreasing and a monotonically increasing function of $\alpha$. Therefore, the MSE has either one minimum or one maximum for $0 < \alpha < \infty$. The derivative of the MSE with respect to $\alpha$ is equal to

$$\frac{\partial E[||e(\alpha)||^2]}{\partial \alpha} = 2 \sum_{l=1}^{N^2} \frac{|D(l)|^2|C(l)|^2(\alpha |\hat{X}(l)|^2|C(l)|^2) - \frac{1}{N^2} \sum_{m=1}^{N^2} |Y(m) - D(m)\hat{X}(m)|^2}{(|D(l)|^2 + \alpha |C(l)|^2)^3}.$$  \hspace{1cm} (31)

Therefore,

$$\frac{\partial E[||e(\alpha)||^2]}{\partial \alpha} < 0, \text{ for } 0 < \alpha < \frac{\sum_{m=1}^{N^2} |Y(m) - D(m)\hat{X}(m)|^2}{N^2 \max_l(|\hat{X}(l)|^2|C(l)|^2)}$$  \hspace{1cm} (32)

and

$$\frac{\partial E[||e(\alpha)||^2]}{\partial \alpha} > 0, \text{ for } \frac{\sum_{m=1}^{N^2} |Y(m) - D(m)\hat{X}(m)|^2}{N^2 \min_l(|\hat{X}(l)|^2|C(l)|^2)} < \alpha < \infty.$$  \hspace{1cm} (33)

Therefore, according to conditions (29), (30), (32) and (33), the MSE function has a unique minimum for $0 < \alpha < \infty$; this minimum is attained by the value $\alpha_{opt}$ which is in the range

$$\frac{\sum_{m=1}^{N^2} |Y(m) - D(m)\hat{X}(m)|^2}{N^2 \max_l(|\hat{X}(l)|^2|C(l)|^2)} < \alpha_{opt} < \frac{\sum_{m=1}^{N^2} |Y(m) - D(m)\hat{X}(m)|^2}{N^2 \min_l(|\hat{X}(l)|^2|C(l)|^2)},$$  \hspace{1cm} (34)

or

$$0 < \delta_{k, opt}(l) \leq \sum_{l=1}^{N^2} \max_l(|X_k(l)|^2|C(l)|^2) - \sum_m |C(m)X_k(m)|^2 = \delta^*_k.$$  \hspace{1cm} (35)

We have derived two conditions for $\delta_k(l)$, condition (35) which is based on an optimality criterion and condition (21) which is based on the convergence analysis. In other words the values of $\delta_k(l)$ used at each iteration, denoted by $\delta_k^{used}(l)$ must satisfy

$$\delta_k^{conv}(l) < \delta_k^{used}(l) < \delta_k^{opt}.$$  \hspace{1cm} (36)

It has been verified experimentally that (36) is usually satisfied. However, for those frequencies for which $D(l)$ is close to zero, it may be that $\delta_k^{conv}(l) > \delta_k^{opt}$. At those frequencies $\delta_k^{used}(l) = \delta_k^{conv}(l)$ does not belong to the optimum range. Clearly for those $\delta_k(l)$ satisfying (36), values in between $\delta_k^{conv}(l)$ and $\delta_k^{opt}$ can also be used.

6. EXPERIMENTAL RESULTS

The performance of the proposed frequency domain iterative restoration algorithm is illustrated with artificially blurred image. Some of these results are presented in this section, where a 256 x 256 pixels portrait image was used, the 2-D Laplacian was used for $C$, the blur used was due to two directional motion over 7x7 pixels, and the criterion $\|x_{k+1} - x_k\|^2/\|x_k\|^2 \leq 10^{-7}$ was used for terminating the iteration. The performance of the restoration algorithm was evaluated by measuring the improvement in signal to noise ratio after k-iterations denoted by $\Delta^{SNR}_k$ and defined by

$$\Delta^{SNR}_k = 10 \log_{10} \frac{\|y - x\|^2}{\|x_k - x\|^2}.$$
To guarantee convergence of iteration (10), the values of $M(l)$ in Eq. (20) need to be determined, so that a value of the parameter $\beta(l)$ is chosen appropriately at every frequency, where $0 < \beta(l) < M(l)$. For the determination of $M(l)$, the minimum value of $\delta_{\text{used}}(l)$ over all iterations, denoted by $\delta_{\text{min}}(l)$, needs to be known. Towards that end, the value of $\delta_{\text{used}}(l)$ estimated from Eq. (21) is set to be equal to $\delta_{\text{min}}(l)$. Based on this value a value of $\beta(l)$ is estimated. If $\delta_{\text{used}}(l) < \delta_{\text{opt}}(l)$ then $\delta_{\text{used}}(l)$ is set to be equal to $\delta_{\text{min}}(l)$ and a new $\beta(l)$ is estimated. It is mentioned here that the convergence analysis presented in Sec. 3.1 does not apply when $\beta(l)$ depends on the iteration index, but only when $\beta(l)$ is kept constant, either for a small number of iterations before $\delta_{\text{min}}(l)$ is reached, or after $\delta_{\text{min}}(l)$ is reached. Therefore, the iteration was rerun with a constant $\beta(l)$ determined by the value of $\delta_{\text{min}}(l)$ (and the corresponding value of $M(l)$ from Eq. (20)) that was found. It was observed in all experiments, that the results obtained the second time the iteration was run were indistinguishable from the ones obtained the first time the iteration was run, and therefore, will not be shown here. More specifically, the $\delta_{\text{min}}(l)$’s obtained in the two cases were almost identical. It is also mentioned here that the value of $\delta_{\text{min}}(l)$ is used only in determining $\beta(l)$, while the actual value of $\delta_{\text{used}}(l)$ is used at each iteration. However, $\delta_{\text{min}}(l)$ can not be analytically estimated from Eq. (21), and it needs to be estimated experimentally. $\delta_{\text{used}}$ and $\delta_{\text{opt}}$ are shown in Figs. 1a and 1b for SNR equal to 20 and 30dB, respectively and $l = (0.16\pi, 0.23\pi)$. The corresponding values of the regularization parameter $\alpha_k$ are shown in Fig. 2.

The values of $\Delta_{\text{SNR}}$ and the required number of iterations, as well as the estimate of the noise variance, $\hat{\sigma}_n^2 = \|y - Dx_k\|^2/N^2$ at convergence, are shown in Table 1 for SNRs of 20, 30 dB. The values of the residual $\|y - D x_k\|^2/N^2$ and the error between iteration steps $\|x_k - x_{k-1}\|^2$, for the two SNRs mentioned above are compared in Figs. 3 and 4 respectively. Note that the plots are scaled appropriately and that the vertical axis in Fig. 4 is logarithmic. It is observed in Fig. 3 that in all cases the residuals converge to a value close to but slightly smaller than the true values. This result is expected, since the solution at convergence (center of an ellipsoid bounding the intersection of $Q_x$ and $Q_{x/y}$ in (2) and (3), respectively) lies inside the ellipsoid $Q_{x/y}$. The plots in Fig. 4 demonstrate that in both cases the iterative algorithm converges with the same rate. In all cases the restored images are very satisfactory, based on the improvement in SNR and visual inspection. The blurred and restored images for this experiment are shown in Fig. 5.

7. CONCLUSIONS

In this paper we have proposed a regularized iterative image restoration algorithm according to which a restored image and an estimate of the regularization matrix are provided simultaneously at each iteration step. Sufficient conditions for the convergence of the algorithm have been derived in the frequency domain. According to this iteration, no knowledge of the noise variance or the bound which determines the ellipsoid which expresses the smoothness in the image is assumed. Linear constraints can also be incorporated into the iteration. When nonlinear constraints are used the linearization step in the analysis of the algorithm can not be applied, although the algorithm has been shown to converge experimentally.
References


Table 1. SNR improvement and estimated noise variance for various SNRs with motion blurred image.

<table>
<thead>
<tr>
<th>SNR</th>
<th>$\Delta_{SNR}$</th>
<th># of iteration</th>
<th>Estimated variance of noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 dB</td>
<td>3.2 dB</td>
<td>45</td>
<td>18.55</td>
</tr>
<tr>
<td>30 dB</td>
<td>4.2 dB</td>
<td>38</td>
<td>2.072</td>
</tr>
</tbody>
</table>

$\sigma_n^2 = 21.60$ at 20 dB and 2.16 at 30 dB

Figure 1. Comparison of $\delta_k^{used}$ and $\delta_k^{opt}$ with 7X7 motion blurred image at $\ell = (0.16\pi, \ 0.23\pi)$ for:

(a) $SNR= 20$ dB and (b) $SNR= 30$ dB.
Figure 2. Values of $\alpha_k$ with 7x7 motion blurred image at $f = (0.16\pi, 0.23\pi)$ and $SNR$ equal to 20dB and 30dB.

Figure 3. Values of $\|y - Dx_k\|^2/N^2$ for various $SNRs$.

Figure 4. Values of $\|x_k - x_{k-1}\|^2$ for various $SNRs$. 
Figure 5. Noisy-blurred images, 7X7 motion blur: (a) $SNR = 20dB$, (c) $SNR = 30dB$. Restored images by the proposed algorithms: (b) of image in (a), $\Delta_{SNR} = 3.2dB$, (d) of image in (c), $\Delta_{SNR} = 4.2dB$. 